JOURNAL OF APPROXIMATION THEORY 63, 368–374 (1990)

A Mergelyan–Vitushkin Approximation Theorem for Rational Modules

JAMES LI-MING WANG

Department of Mathematics, University of Alabama, Tuscaloosa, Alabama 35487

Communicated by O. Shisha

Received July 2, 1987

1. INTRODUCTION

Let X be a compact subset of the complex plane $\mathbb{C}$ and let g be a continuous function on X. We denote by $\mathscr{R}(X, g)$ the rational module

$$\{r_0(z) + r_1(z)\, g(z)\},$$

where each r_i is a rational function with poles off X.

In the case in which $g(z) = \bar{z}$, the closures of $\mathscr{R}(X, \bar{z})$ in various norms were first considered by O'Farrell [4]. Later, several authors (e.g., Carmona, Trent, Verdera, and Wang) explored the subject. A question which arose from these investigations concerned the characterization of $R(X, g)$, the uniform closure of $\mathscr{R}(X, g)$ in $C(X)$. When X has empty interior $\mathring{X}$, this was settled in [6] (also see [1]) by showing that $R(X, g) = C(X)$ if and only if $R(Z) = C(Z)$, where g is a smooth function and Z is the subset of X on which $\bar{\partial} g$ vanishes. Here $\bar{\partial} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ is the usual Cauchy–Riemann operator in the complex plane.

The existence of interior points, however, makes the problem more difficult. Note that a function f satisfies $\bar{\partial}(\bar{\partial} f/\bar{\partial} g) = 0$ in an open set U if and only if $f = h + gk$ with h and k holomorphic. Therefore it is natural to ask the following question: For an arbitrary compact set X, is

$$R(X, g) = \{f \in C(X) : \bar{\partial}(\bar{\partial} f/\bar{\partial} g) = 0 \text{ in } \mathring{X}\}$$

whenever $\bar{\partial} g \neq 0$ on X? In particular, when $g(z) = \bar{z}$, this would be viewed as an approximation problem for the elliptic differential operator $\bar{\partial}^2 = \bar{\partial} \circ \bar{\partial}$: Is

$$R(X, \bar{z}) = \{f \in C(X) : \bar{\partial}^2 f = 0 \text{ in } \mathring{X}\} \qquad (*)$$

for an arbitrary compact set X?

For the case when X is a compact set whose complement is connected, the approximation problem is not too difficult. In [1], the method used by Mergelyan [3] is extended by Carmona to obtain a positive result for

0021-9045/90 $3.00
Copyright © 1990 by Academic Press, Inc.
All rights of reproduction in any form reserved.

question (∗). Actually the Mergelyan lemma can be extended so that (∗) is also true for any compact set X satisfying the following capacity condition (see [1]):

$$\gamma(\Delta(z, r) - X) \geqq cr$$

for some positive constant c, for every point z on the boundary of X, and for all sufficiently small r, where γ is the analytic capacity [2]. In particular, this condition is satisfied if the diameters of the components of the complement of X are bounded away from zero. However, the general case remains unknown. The theory of approximation by rational modules does not follow directly from that of rational functions. A simple example shows that there is a continuous function $f \in C(X)$ with $f = h + \bar{z}k$, so that h and k are holomorphic in $\mathring{X}$ but unbounded. So far we do not know an example of a compact set X in which (∗) is not true.

The inner boundary of X is the set of the boundary points of X not belonging to the boundary of a component of the complement of X. In this paper, we employ a scheme for approximation used by Vitushkin [7] together with the extended Mergelyan lemma [1] to prove the following:

THEOREM. *If the inner boundary of a compact set X is empty, then* $R(X, \bar{z}) = \{f \in C(X) : \bar{\partial}^2 f = 0 \text{ in } \mathring{X}\}$.

With localization argument [8] we obtain the following:

COROLLARY. *If the inner boundary of a compact set X is at most countable, then* $R(X, \bar{z}) = \{f \in C(X) : \bar{\partial}^2 f = 0 \text{ in } \mathring{X}\}$.

We establish now some additional general notations. We denote by m the Lebesgue measure on $\mathbb{C}$ and by $\Delta(z, r)$ the open disc with center z and radius r. If U is an open subset of $\mathbb{C}$, then we denote by $C_c^i(U)$, $i = 0, 1, 2, \ldots$, the space of i-times continuously differentiable complex functions on U with compact support and by $H(U)$ the space of holomorphic functions in U. If μ is a compactly supported Borel measure on $\mathbb{C}$, we write $\hat{\mu}(z) = \int d\mu(\xi)/(\xi - z)$ for the Cauchy transform of μ and $\tilde{\mu}(z) = \int (\bar{\xi} - \bar{z})/(\xi - z)\, d\mu(\xi)$. If $\phi \in L^1(m)$ has compact support, then we write $\hat{\phi} = (\phi m)^\wedge$ and $\tilde{\phi} = (\phi m)^\sim$. The symbol c stands for a positive constant, independent of the relevant variables under consideration (unless otherwise stated) and not necessarily the same at each occurrence.

2. PROOF OF THE THEOREM

While the following lemma is well known for certain more general elliptic differential operators [5], its proof for the operator $\bar{\partial}^2$ is surprisingly simple (cf. [1]).

LEMMA 1. *Let K be a compact subset of an open set U in $\mathbb{C}$. Then there exist a compact set $\tilde{K}$ such that $K \subsetneq \tilde{K} \subset U$ and a constant $c = c(K, U)$ with the following property: If $\bar{\partial}^2 f = 0$ in U, then $|\bar{\partial} f(z)| \leqslant c \, \|f\|_{\infty, \tilde{K}}$ for all $z \in K$, where $\|f\|_{\infty, \tilde{K}}$ is the* sup norm *of f on the compact set $\tilde{K}$.*

Proof. As noted in the Introduction, we can write $f = h + \bar{z}k$, with $h, k \in H(U)$ so that $k = \bar{\partial} f$. Let $0 < 2r < \text{distance}(K, \mathbb{C} \backslash U)$. Then for each $z_0 \in K$ we obtain from Green's theorem

$$k(z_0) = \frac{c}{r^2} \int_{\Delta(z_0; r)} \bar{\partial} f(z) \, dm(z) = \frac{c}{r^2} \int_{|z - z_0| = r} f(z) \, dz$$

and the lemma is proved if we take $\tilde{K} = \{z \in U: \text{distance}(z, K) \leqslant r\}$.

Let f be a continuous function on $\mathbb{C}$, and $\phi \in C_c^2(\mathbb{C})$. We define the (Vitushkin) localization operator V_ϕ by

$$V_\phi(f) = f \cdot \phi + \frac{2}{\pi} (f \cdot \bar{\partial} \phi)^\wedge + \frac{1}{\pi} (f \cdot \bar{\partial}^2 \phi)^\sim.$$

$V_\phi(f)$ is again continuous. Furthermore we may use the operator V_ϕ to "localize" the singularities of f because $\bar{\partial}^2 V_\phi(f) = \phi \bar{\partial}^2 f$ in the sense of distributions.

The proof of the following lemma can be found in [8].

LEMMA 2. *Let $f \in C_c(\mathbb{C})$ and $\phi \in C_c^2(\Delta(z_0; \delta))$. Then $\|V_\phi(f)\|_\infty \leqslant c\omega(f; \delta)(\|\phi\|_\infty + \|\bar{\partial}\phi\|_\infty \cdot \delta + \|\bar{\partial}^2 \phi\|_\infty \cdot \delta^2)$, where $\omega(f; \delta)$ is the modulus of continuity of f and $\| \ \|_\infty$ is the usual* sup norm *on the plane $\mathbb{C}$.*

Fix $\delta > 0$. We consider the special system of partitions of unity $\{\Delta_j, \phi_j\}$ used by Vitushkin [7]:

(a) The family $\{\Delta_j\}$ with $\Delta_j = \Delta(z_j, \delta)$ is an almost disjoint countable covering of the plane. This means that each z in the plane belongs to Δ_j for at most c_0 indices j. Here c_0 is an absolute constant ($c_0 = 21$, say).

(b) $\phi_j \in C_c^\infty(\Delta_j)$, $0 \leqslant \phi_j \leqslant 1$, $\sum \phi_j = 1$, $|\nabla \phi_j| \leqslant c/\delta$, and $|\nabla^2 \phi_j| \leqslant c/\delta^2$.

Let $f \in C_c(\mathbb{C})$. We fix $\delta > 0$ and write $f = \sum f_j$ as a finite sum, where $f_j = V_{\phi_j}(f)$. We conclude from Lemma 2 that

$$\|f_j\|_\infty \leqslant c\omega(f; \delta) \qquad \text{for each } j.$$

The following lemma, proved in [1], is analogous to a result due to Mergelyan [3], essential in the original proof of his theorem on uniform approximation by rational functions.

LEMMA 3. *Let D be an open disc of radius $r>0$. Let E be a compact, connected subset of D with diameter at least r and such that $\Omega=(\mathbb{C}-E)\cup\{\infty\}$ is connected. Then there is a constant $c>0$ such that, for every $\xi\in\mathbb{C}$, there exist $H(\xi,\cdot)\in H(\Omega)$ and $Q(\xi,\cdot)\in H(\Omega)+\bar{z}H(\Omega)$ satisfying*

(a) $|H(\xi,z)|\leqslant c/r, \qquad |Q(\xi,z)|\leqslant c,$

(b) $\left|H(\xi,z)-\frac{1}{z-\xi}\right|\leqslant cr^3/|z-\xi|^4, \qquad \left|Q(\xi,z)-\frac{\bar{z}-\bar{\xi}}{z-\xi}\right|\leqslant cr^3/|z-\xi|^3,$

for $z\in\Omega$, $\xi\in D$, and $z\neq\xi$.

The $Q(\xi,z)$ in Lemma 2 can be taken as $(\bar{z}-\bar{\xi})\,H(\xi,z)$.

LEMMA 4. *Let $f\in C_c(\mathbb{C})$ with $\bar{\partial}^2 f=0$ in $\mathring{X}$. Let B be a compact subset of ∂X such that $\Delta(z,\delta)\backslash X$ contains a compact connected subset with diameter at least $c\delta$ for all $z\in B$, all $\delta\leqslant\delta_0$, where $\delta_0>0$, $c>0$ do not depend on z. For any fixed $0<\delta\leqslant\delta_0/4$, there is a continuous function $G(z)$ with the following properties*:

$$\|f-G\|_\infty\leqslant c\omega(f;\delta)$$

and $\bar{\partial}^2 G=0$ in $\mathring{X}$ and in some open set containing B.

Proof. We fix $\delta\leqslant\delta_0/4$ and write $f(z)$ as $f(z)=\sum f_j(z)$, where $f_j=V_{\phi_j}(f)$. Let j be an index such that there is a point $z_j\in B$ for which the distance $d(\Delta_j,z)\leqslant\delta$. Applying Lemma 3, we obtain functions $H_j(\xi,\cdot)\in H(\Omega_j)$, $Q_j(\xi,\cdot)\in H(\Omega_j)+\bar{z}H(\Omega_j)$, where Ω_j is an open set containing X and $\mathbb{C}-\Omega_j$ is a compact subset of $\Delta(z_j,4\delta)\backslash X$, satisfying

$$|H_j(\xi,z)|\leqslant c/\delta, \qquad |Q_j(\xi,z)|\leqslant c,$$

$$\left|H_j(\xi,z)-\frac{1}{z-\xi}\right|\leqslant c\delta^3/|z-\xi|^4,$$

$$\left|Q_j(\xi,z)-\frac{\bar{z}-\bar{\xi}}{z-\xi}\right|\leqslant c\delta^3/|z-\xi|^3,$$

for all $z\in\Omega_j$ and $\xi\in\Delta(z_j,4\delta)$. We put

$$G_j(z)=-\frac{2}{\pi}\int[f(\xi)-f(z_j)]\,\bar{\partial}\phi_j(\xi)\,H_j(\xi,z)\,dm(\xi)$$
$$+\frac{1}{\pi}\int[f(\xi)-f(z_j)]\,\bar{\partial}^2\phi_j(\xi)\,Q_j(\xi,z)\,dm(\xi)$$

so that $G_j \in H(\Omega_j) + \bar{z}H(\Omega_j)$. Now suppose that j is such that the distance $d(\Delta_j, B) \geqslant \delta$. In this case we put $G_j = f_j$ so that $\bar{\partial}^2 G_j = 0$ in $\mathring{X}$ and in a δ-neighborhood of B.

Since $f_j = V_{\phi_j}(f) = V_{\phi_j}(f - f(z_j))$, we have

$$\begin{aligned}
|f_j(z) - G_j(z)| &\leqslant |[f(z) - f(z_j)]\, \phi_j(z)| \\
&\quad + \frac{2}{\pi} \int |f(\xi) - f(z_j)| \cdot |\bar{\partial}\phi_j(\xi)| \cdot \left| H_j(\xi, z) - \frac{1}{z - \xi} \right| dm(\xi) \\
&\quad + \frac{1}{\pi} \int |f(\xi) - f(z_j)| \cdot |\bar{\partial}^2 \phi_j(\xi)| \cdot \left| Q_j(\xi, z) - \frac{\bar{z} - \bar{\xi}}{z - \xi} \right| dm(\xi) \\
&\leqslant \omega(f;\delta) |\phi_j(z)| + c\, \frac{\omega(f;\delta)}{\delta} \int_{\Delta_j} \frac{\delta^3}{|z - \xi|^4}\, dm(\xi) \\
&\quad + c\, \frac{\omega(f;\delta)}{\delta^2} \int_{\Delta_j} \frac{\delta^3}{|z - \xi|^3}\, dm(\xi) \\
&\leqslant \omega(f;\delta) |\phi_j(z)| + c\omega(f;\delta)\, \delta^2 \int_{\Delta_j} \frac{dm(\xi)}{|z - \xi|^4} \\
&\quad + c\omega(f;\delta)\, \delta \int_{\Delta_j} \frac{dm(\xi)}{|z - \xi|^3}
\end{aligned}$$

for all z.

Let $G(z) = \sum G_j(z)$. It is clear that $\bar{\partial}^2 G = 0$ in $\mathring{X}$ and in some neighborhood containing B. We are going to estimate $\sum |f_j(z) - G_j(z)|$.

First we note that $\|f_j - G_j\|_\infty \leqslant \|f_j\|_\infty + \|G_j\|_\infty \leqslant c\omega(f;\delta)$ by Lemma 2 and Lemma 3. Hence from property (a) of the system $\{\Delta_j, \phi_j\}$, we have

$$\begin{aligned}
\sum |f_j(z) - G_j(z)| &\leqslant \sum_{|z_j - z| \leqslant 4\delta} |f_j(z) - G_j(z)| \\
&\quad + \sum_{|z_j - z| > 4\delta} |f_j(z) - G_j(z)| \\
&\leqslant c\omega(f;\delta) + c\omega(f;\delta)\, \delta^2 \int_{|\xi - z| \geqslant \delta} \frac{dm(\xi)}{|z - \xi|^4} \\
&\quad + c\omega(f;\delta)\, \delta \int_{|\xi - z| \geqslant \delta} \frac{dm(\xi)}{|z - \xi|^3} \\
&\leqslant c\omega(f;\delta).
\end{aligned}$$

Remark. In the hypothesis of Lemma 4, if $\bar{\partial}^2 f = 0$ also on some open set U in $\mathbb{C}$, then by substituting $d(\Delta_j, B)$ with $d(\Delta_j, B \backslash U)$ in the above construction, one can show that the resulted continuous function G satisfies the additional condition $\bar{\partial}^2 G = 0$ in U. This observation is used in the proof of the following lemma.

LEMMA 5. *Let $f \in C_c(\mathbb{C})$ with $\bar{\partial}^2 f = 0$ in $\mathring{X}$. Let B_n be a subset of ∂X such that $\Delta(z, \delta)\backslash X$ contains a compact connected subset with diameter at least δ/n for all $z \in B_n$, all $\delta < 1/n$. Then for any $\varepsilon > 0$, there is a continuous function $G(z)$ with the following properties*:

$$\|f - G\|_\infty \leqslant \varepsilon$$

and $\bar{\partial}^2 G = 0$ in $\mathring{X}$ and in some neighborhood of UB_n.

Proof. For any $z \in \overline{B_n}$, the closure of B_n, $\Delta(z, \delta)\backslash X$ contains a compact connected subset with diameter at least $\delta/2n$ for all $\delta \leqslant 1/n$. We put $G_0 = f$. By Lemma 4, we can construct a sequence of continuous functions G_n with the following properties: $\|G_n - G_{n-1}\| \leqslant \varepsilon/2^n$ and $\bar{\partial}^2 G_n = 0$ in $\{\bar{\partial}^2 G_{n-1} = 0\}$ and in some neighborhood of $\overline{B_n}$. Let $G(z) = \lim_{n \to \infty} G_n(z)$. Then $G(z)$ is the desired function from Lemma 1.

Proof of Theorem. Let z belong to the boundary of a component Ω_i of the complement of X with diameter $d(\Omega_i) = d_i$. For $\delta < d_i$, $\Delta(z, \delta)\backslash X$ contains a compact connected subset with diameter at least δ. The Theorem now follows from Lemma 5.

3. REMARKS

1. Lemma 2 through 5 are good if we use the notion of analytic capacity or continuous analytic capacity [2] of the set $\Delta(z, \delta)\backslash X$ instead of the diameter. For a matter of simplicity, we choose to avoid them.

2. The argument used in this paper can be extended to certain general rational modules and to certain elliptic differential operators.

REFERENCES

1. J. J. CARMONA, Mergelyan's approximation theorem for rational modules, *J. Approx. Theory* **44** (1985), 113–126.
2. T. W. GAMELIN, "Uniform Algebras," Prentice–Hall, Englewood Cliff, NJ, 1969.
3. S. N. MERGELYAN, Uniform approximation to functions of complex variables, *Amer. Math. Soc. Transl.* **101** (1954), 294–391.

4. A. G. O'Farrell, Annihilators of rational modules, *J. Funct. Anal.* **19** (1975), 373–389.
5. G. E. Silov, "Generalized Functions and Partial Differential Equations," Gordon & Breach, New York, 1968.
6. T. T. Trent and J. L. Wang, The uniform closure of rational modules, *Bull. London Math. Soc.* **13** (1981), 415–420.
7. A. G. Vitushkin, Analytic capacity of sets and problems in approximation theory, *Russian Math. Surveys* **22** (1967), 139–200.
8. J. L. Wang, A localization operator for rational modules, *Rocky Mountain J. Math.* **19** (1989), 999–1002.